GENERATING OPERATORS OF THE
KRASIL'SHCHIK-SCHOUTEN BRACKET

JOSÉ A. VALLEJO

ABSTRACT. It is proved that given a divergence operator on the structural sheaf of graded commutative algebras of a supermanifold, it is possible to construct a generating operator for the Krashil'shchik-Schouten bracket. This is a particular case of the construction of generating operators for a special class of bigraded Gerstenhaber algebras.

Also, some comments on the generalization of these results to the context of $n-$graded Jacobi algebras are included.

1. INTRODUCTION

These notes intend to describe, in a relatively self-contained manner, the construction of a generating operator for the Krasil'shchik-Schouten bracket on the multiderivations of a supermanifold $(M, \mathcal{A})$, $\mathcal{D}(\mathcal{A})$, once a divergence operator on the structural sheaf $\mathcal{A}$ is given. For simplicity, we do not study the more general case, in which $\mathcal{A}$ is a sheaf of $n-$graded commutative algebras, but we restrict ourselves to the common situation in supermanifold theory of $\mathbb{Z}-$graded commutative algebras (that is, $n = 1$). Nevertheless, the changes needed to deal with the general case are mainly of notational character.

Thus, we answer affirmatively a conjecture raised by Kosmann-Schwarzbach and Monterde in [Kos-Mon 02]. To put it in context, let us recall that the notion of generating operator (and related Gerstenhaber algebras) has its origins in the quantization of non abelian gauge theories through the Batalin-Vilkovisky (BV) formalism (see [Bat-Vil 81], [Kos 95b], [Wit 90]). Concretely, Witten showed in [Wit 90] (see also [Sch 93]) that the master equation and the condition of nilpotency of the BV operator have an algebraic background: they make sense each time one has a Gersternhaber algebra with generating operator of square zero. This is the case, for example, of an odd Poisson structure and the associated Hamiltonian divergence; a detailed study of these structures is presented in [Kos-Mon 02], where the analogous results in the graded case (the Krasil'shchik-Schouten bracket) are left as a conjecture.

The first part of the paper contains a brief review of the basic notions of supermanifold theory needed to understand the construction of the Krasil'shchik-Schouten bracket on multiderivations, which is an abstract generalization of the graded Schouten bracket on graded multivectors. After presenting Krasil'shchik work, we introduce the notion of divergence operator, a graded reformulation of the usual divergence in manifold theory. It is shown how to extend such a divergence to a differential operator on the algebra of multiderivations $\mathcal{D}(\mathcal{A})$, through a recursive algorithm, and we prove that the result is a generating operator of the Krashil'shchick-Schouten bracket on $\mathcal{D}(\mathcal{A})$. All the construction is based on the fact that the algebra of multiderivations endowed with the Krashil'shchick-Schouten bracket, $(\mathcal{D}(\mathcal{A}), [\![\_, \_]\!])$, is a bigraded Gerstenhaber algebra. The procedure to define such generating operators is well known in the $\mathbb{Z}$−graded case, and goes back to the work of Koszul [Koz 85] (see also [Kos 95] for a study in the context of Lie algebroids); it turns out that the same ideas can be applied in the bigraded context.

Finally, we make some digressions about the generalization of these results to the case in which one deals with an n−graded Jacobi algebra, where Leibniz's rule does not hold.

All the gradings, unless otherwise specified, will be understood as $\mathbb{Z}$−gradings.

2. Some results from Supermanifold Theory

2.1. **Basic definitions.** For references, see [Kos 77] or [Lei 80]. Let K be the commutative field $\mathbb{R}$ or $\mathbb{C}$, and M a differential manifold. The set of differentiable functions on M gives us an example of sheaf $\mathcal{F}$; if U is open in M, we have

$$\mathcal{F}(U) = \{f : U \to \mathbb{R} : f \in C^\infty(U)\} \equiv C^\infty(U)$$

and, for $V \subset U$ open,

$$\rho_V^U : C^\infty(U) \to C^\infty(V)$$
$$f \mapsto f|_V.$$

Indeed, this is a sheaf of graded commutative algebras, although in this case the grading is trivial. The basic idea underlying the definition of a graded manifold, is the substitution of the commutative sheaf of differentiable functions by another one in which we can accommodate objects with a $\mathbb{Z}_2$−grading.

Definition 1. *A graded manifold (or supermanifold) of dimension $(m|n)$ and basis $(M, C^\infty(M))$ is given by a usual differential manifold M, with dimension m, and a sheaf $\mathcal{A}$ of graded K−commutative algebras (the structural sheaf) such that*

(1) *We have an exact sequence of sheaves*

$$0 \to \mathcal{N} \to \mathcal{A} \xrightarrow{\sim} C^\infty(M) \to 0, \tag{2.1}$$

where $\mathcal{N}$ is the sheaf of nilpotents of $\mathcal{A}$ and $\sim$ is a surjective morphism of graded K−commutative algebras.

(2) *$\mathcal{N}/\mathcal{N}^2$ is a locally free module with rank n over $C^\infty(M) = \mathcal{A}/\mathcal{N}$, and $\mathcal{A}$ is locally isomorphic, as a sheaf of graded K−commutative algebras, to the exterior bundle $\Lambda_{C^\infty(M)}(\mathcal{N}/\mathcal{N}^2)$.*

Remark 1. *In the following, we will consider only real graded manifolds, that is, those for which $K = \mathbb{R}$. Thus, we will omit the corresponding prefix.*

From the exact sequence (2.1) we obtain, for any open $U \subset X$, the exact sequence of graded algebras

$$0 \to \mathcal{N}(U) \to \mathcal{A}(U) \overset{\sim}{\to} C^\infty(U) \to 0.$$

A section f of $\mathcal{A}$ will be called a graded function (or superfunction). The image of such a graded function $f \in \mathcal{A}(U)$ by the structural morphism $\sim$ is denoted $\tilde{f}$.

Next, we introduce morphisms between graded manifolds.

Definition 2. *A morphism of graded manifolds is a pair of mappings* (f, ϕ) : $(M, \mathcal{A}) \to (N, \mathcal{B})$ *where* $f : M \to N$ *is a differential mapping between usual manifolds and for each open* $U \subset N$, $\phi : \mathcal{B}(U) \to \mathcal{A}(f^{-1}(U))$ *is an even morphism of graded algebras compatible with the restrictions, and all such that the diagram*

$$
\begin{array}{ccc}
\mathcal{B}(U) & \xrightarrow{\ \ \phi\ \ } & \mathcal{A}(f^{-1}(U)) \\
\downarrow & & \downarrow \\
C^\infty(U) & \xrightarrow{\ \ f_*\ \ } & f_* C^\infty(f^{-1}(U))
\end{array}
$$

commutes.

Isomorphisms, as a particular case, are defined in an obvious way.

According to the definition, the structural sheaf of a supermanifold $(M, \mathcal{A})$ (with dimension $(m|n)$) is locally isomorphic to the locally free $C^\infty(M)-$module $\mathcal{N}/\mathcal{N}^2$, which has rank n, so by the Serre duality theorem (see [Ser 55]) there exists a vector bundle of rank n over M, E, such that any point of M has an open neighborhood U for which $\mathcal{A}(U) \simeq \Gamma_U(\Lambda E)$ as graded commutative $\mathbb{R}-$algebras. The splitting neighborhoods of a supermanifold are those for which these two representations of the structural sheaf are identified.

Definition 3. *A splitting neighborhood of a supermanifold* $(M, \mathcal{A})$ *is an open* $U \subset M$ *such that* $E|_U$ *is a trivial bundle and* $\mathcal{A}|_U \simeq \Lambda_{C^\infty(U)}(\mathcal{N}/\mathcal{N}^2)$.

If U is a splitting neighborhood for $(M, \mathcal{A})$ there exists a basis, which we will denote $\{x^{-1}, ..., x^{-n}\}$, of sections of $E|_U$, along with an isomorphism

$$\mathcal{A}(U) \simeq C^\infty(U) \otimes_{\mathbb{R}} \Lambda E_n$$

where $E_n = \langle \{x^{-1}, ..., x^{-n}\} \rangle$ is the vector $\mathbb{R}-$space generated by $\{x^{-1}, ..., x^{-n}\}$. Also, there exists a section $\sigma : C^\infty(U) \hookrightarrow \mathcal{A}(U)$ for the structural morphism $\sim$.

Definition 4. *If* U *is a splitting neighborhood, a family* $\{x^i, x^{-j}\}_{1 \leq i \leq m}^{1 \leq j \leq n}$ *of superfunctions* $(|x^i| = 0, |x^{-j}| = 1)$ *is called a graded coordinate system (or supercoordinate system) if*

(1) $x^i = \sigma(\tilde{x}^i)$ $(1 \leq i \leq m)$, *where* $\{\tilde{x}^1, ..., \tilde{x}^m\}$ *is an ordinary coordinate system on* U.

(2) $\{x^{-1}, ..., x^{-n}\}$ *is a basis of sections of* $E|_U$, *that is,* $x^{-1}, ..., x^{-n} \in \Lambda E_n$ *and* $\prod_{j=1}^{n} x^{-j} \neq 0$.

Remark 2. *As we have already mentioned, the structural sheaf of a supermanifold* $(M, \mathcal{A})$ *is locally isomorphic to* $\Lambda_{C^\infty(M)}(\mathcal{N}/\mathcal{N}^2)$. *An important Theorem (known as the Batchelor theorem [Bat 79], but see also [Gaw 77]), guarantees that this is*

not true just locally, but also globally. Thus, for any supermanifold $(M, \mathcal{A})$ there exists a vector bundle over M, E, such that the sheaf $\mathcal{A}$ can be identified with the sheaf of sections of the exterior bundle associated to E. However, this isomorphism is not canonical.

2.2. Graded vector fields and differential forms.

Definition 5. *Let $\mathcal{F}, \mathcal{G}$ be sheaves on a topological space X. For any open $U \subset M$, let $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ denote the space of morphisms between the restricted sheaves $\mathcal{F}|_U$ and $\mathcal{G}|_U$. The sheaf of homomorphisms is the sheaf $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ given by $\mathrm{Hom}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ with the natural restriction morphisms.*

Let us consider now a graded manifold $(M, \mathcal{A})$, where $\mathcal{A}$ is a sheaf of graded commutative $\mathbb{R}$−algebras. Write $\mathrm{End}_{\mathbb{R}}(\mathcal{A}) = \mathrm{Hom}_{\mathbb{R}}(\mathcal{A}, \mathcal{A})$ for the sheaf of $\mathbb{R}$−linear endomorphisms.

Definition 6. *The sheaf of derivations of $(M, \mathcal{A})$ is the subsheaf of $\mathrm{End}_{\mathbb{R}}(\mathcal{A})$ whose sections on an open subset $U \subset M$ are $\mathbb{R}$−linear graded derivations $D : \mathcal{A}|_U \to \mathcal{A}|_U$. This sheaf is denoted $\mathrm{Der}_{\mathbb{R}}(\mathcal{A})$ or simply $\mathrm{Der}(\mathcal{A})$, and its elements are called the graded vector fields (or supervector fields) of the graded manifold $(M, \mathcal{A})$. Sometimes, we will write $\mathrm{Der}(\mathcal{A}) = \mathcal{X}_G(M)$.*

Let U be a coordinate neighborhood for a graded manifold $(M, \mathcal{A})$ with graded coordinates $\{x^i, x^{-j}\}_{1 \leq i \leq m}^{1 \leq j \leq n}$. There exist even derivations $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^m}$ and odd derivations $\frac{\partial}{\partial x^{-1}}, ..., \frac{\partial}{\partial x^{-m}}$ of $\mathcal{A}(U)$ uniquely characterized by the conditions

$$\tfrac{\partial x^j}{\partial x^i} = \delta_i^j; \quad \tfrac{\partial x^{-j}}{\partial x^i} = 0; \quad \tfrac{\partial x^j}{\partial x^{-i}} = 0; \quad \tfrac{\partial x^{-j}}{\partial x^{-i}} = \delta_i^j$$

(negative indices run from $-n$ to -1, positive ones from 1 to m) and such that every derivation $X \in \mathrm{Der}(\mathcal{A}(U))$ can be written as

$$X = \sum_{i=1}^{m} X(x^i) \frac{\partial}{\partial x^i} + \sum_{j=1}^{m} X(x^{-j}) \frac{\partial}{\partial x^{-j}},$$

where $X(f)$ denotes the acion of X on a superfunction. In particular, $\mathrm{Der}(\mathcal{A}(U))$ is a free right $\mathcal{A}(U)$−module with basis $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^m}; \frac{\partial}{\partial x^{-1}}, ..., \frac{\partial}{\partial x^{-m}}$.

If $U \subset M$ is an open set, the algebraic dual of the graded $\mathcal{A}(U)$−module $\mathrm{Der}(\mathcal{A}(U))$ is $\mathrm{Der}^*(\mathcal{A}(U)) = \mathrm{Hom}_{\mathcal{A}}(\mathrm{Der}(\mathcal{A}(U)), \mathcal{A}(U))$, which has itself a natural structure of graded $\mathcal{A}(U)$−module and we get then a sheaf $U \to \mathrm{Der}^*(\mathcal{A}(U))$. On $\mathrm{Der}^*(\mathcal{A})$ we can introduce an exterior product in the usual way, obtaining the following.

The sheaves of graded homogeneous differential forms on $(M, \mathcal{A})$ are the sheaves

$$\Omega_G^p(\mathcal{A}) = \Lambda^p \mathrm{Der}^*(\mathcal{A}).$$

We will put $\Omega_G(\mathcal{A}) = \sum_{p \in \mathbb{N}} \Omega_G^p(\mathcal{A})$ (also, we understand $\Omega_G^0(\mathcal{A}) = \mathcal{A}$).

The graded differential forms on $(M, \mathcal{A})$ will be sometimes called graded forms.

Being a graded homomorphism of graded modules, a graded differential form has a degree. Thus, we can define a $\mathbb{Z} \times \mathbb{Z}$−bigrading on the module of graded differential forms and we will say that a graded homogeneous differential form λ has bidegree $(p, k) \in \mathbb{Z} \times \mathbb{Z}$ (denoted $\lambda \in \Omega_G^{(p,k)}(\mathcal{A})$) if

$$\lambda : \mathrm{Der}(\mathcal{A}) \times .^{p)}. \times \mathrm{Der}(\mathcal{A}) \longrightarrow \mathcal{A}$$

and if, for all $X_1, ..., X_p \in \mathrm{Der}(\mathcal{A})$,

$$|\langle X_1, ..., X_p; \lambda \rangle| = \sum_{i=1}^{p} |X_i| + k.$$

Thus, we have $\Omega_G^p(\mathcal{A}) = \sum_{k \in \mathbb{Z}} \Omega_G^{(p,k)}(\mathcal{A})$.

Remark 3. *In order to avoid confusions when speaking about bigraded objects, we adopt the convention of naming "cohomological degree" the degree which indicates the number of arguments that the object admits (p above), and "ghost degree" the other one (this terminology comes from physics, more precisely from the quantization of non abelian gauge theories).*

Using this bigrading, any arbitrary (of non homogeneous ghost degree) graded $p-$differential form λ can be decomposed as a sum $\lambda = \lambda_{(0)} + ... + \lambda_{(n)}$, where $\lambda_{(i)}$ is an homogeneous graded form of bidegree (p, i).

The insertion operator is defined as usual. If λ is a graded form of bidegree (p, k) and X is a derivation of degree $|X|$, then $\iota_X \lambda$ is the graded form of bidegree $(p - 1, k + |X|)$ defined by

$$\langle X_1, \ldots, X_{p-1}; \iota_D \lambda \rangle := \langle X_1, \ldots, X_{p-1}, X; \lambda \rangle.$$

This implies that the bidegree of the operator $\iota(X) = \iota_X$ is $(-1, |X|)$.

We shall denote by d^G the graded exterior differential. (See [Kos 77] for details.) In particular, for a graded 0-form $\alpha \in \mathcal{A}$, $\langle X; \mathrm{d}^G \alpha \rangle = X(\alpha)$, and for a graded 1-form, λ, on $(M, \mathcal{A})$ we have

$$\langle X_1, X_2; (\mathrm{d}^G \lambda) \rangle = X_1(\langle X_2; \lambda \rangle) - (-1)^{|X_1||X_2|} X_2(\langle X_1; \lambda \rangle) - \langle [X_1, X_2]; \lambda \rangle.$$

The graded exterior differential is an operator of bidegree $(1, 0)$.

Other familiar operators on ordinary manifolds also have counterparts on graded manifolds. If $X \in \mathrm{Der}(\mathcal{A})$, then the Lie operator $\mathcal{L}_X^G$ is defined by

$$\mathcal{L}_X^G = \iota_X \circ \mathrm{d}^G + \mathrm{d}^G \circ \iota_X. \tag{2.2}$$

Note that $\mathcal{L}_X^G$ is a derivation of bidegree $(0, |X|)$.

2.3. **Graded multivector fields.** We have introduced in the preceding Section the concept of graded vector field; now, we study graded multivector fields.

Let $\Omega_G^p(\mathcal{A}) = \sum_{k \in \mathbb{Z}} \Omega_G^{(p,k)}(\mathcal{A})$. A graded $q-$multivector field A is a $C^\infty(M)-$ multilinear alternating morphism of sheaves

$$A : \Omega_G^1(\mathcal{A}) \times \overset{q)}{...} \times \Omega_G^1(\mathcal{A}) \longrightarrow \mathcal{A},$$

and, just as in the case of graded differential forms, the graded multivector fields are actually bigraded objects. We will say that a graded multivector field A has bidegree $(A_1, A_2) \in \mathbb{Z} \times \mathbb{Z}$ (and will denote it $A \in \Gamma(\Lambda_G^{(A_1,A_2)}(\mathcal{A}))$ or $A \in A_G^{(A_1,A_2)}(\mathcal{A})$) if

$$|A(\lambda_1, ..., \lambda_{A_1})| = \sum_{i=1}^{q} |\lambda_i| + A_2;$$

also we denote $\Gamma(\Lambda_G^q(\mathcal{A})) = \sum_{k \in \mathbb{Z}} \Gamma(\Lambda_G^{(q,k)}(\mathcal{A}))$.

Graded multivector fields are generated by decomposable elements of the form

$$A = X_1 \wedge ... \wedge X_q \in \Gamma(\Lambda_G^q(\mathcal{A})),$$

where $X_i \in \mathrm{Der}(\mathcal{A})$, $1 \le i \le q$.

Associated to each $A \in \Gamma(\Lambda_G^q(\mathcal{A}))$, we have the insertion operator ι_A, which is defined in analogy with the corresponding operator for graded vector fields; if A is a decomposable element as above, we have

$$\iota_A : \Omega_G^p(\mathcal{A}) \to \Omega_G^{p-q}(\mathcal{A})$$
$$\varsigma \mapsto \iota_{X_q} \circ ... \circ \iota_{X_1}(\varsigma).$$

Note that ι_A is an operator on $\Omega_G(\mathcal{A}) = \sum_{p \in \mathbb{Z}} \Omega_G^p(\mathcal{A})$ with bidegree $(-q, \sum_{i=1}^{q} X_i)$ (for ease in writing, we will omit the bars when denoting degrees, if there is no risk of confusion). In the same way, the operator $\mathcal{L}_A^G = \iota_A \circ \mathrm{d}^G + \mathrm{d}^G \circ \iota_A$ has bidegree $(-q+1, \sum_{i=1}^{q} X_i)$. It generalizes the graded Lie derivative (2.2).

2.4. The graded Schouten-Nijenhuis bracket.
We have already found several examples of differential operators. Let us consider them in more detail.

Remark 4. *In the following, we will particularize to differential operators on $\Omega_G(\mathcal{A})$, because this is the main example we will use. But all the definitions and properties remain unchanged, with the obvious modifications in notation, if we consider a general bigraded (or $n-$graded) algebra $\mathcal{B}$ instead of $\Omega_G(\mathcal{A})$.*

Let $\mathrm{End}_\mathbb{R}(\Omega_G(\mathcal{A}))$ be the space of $\mathbb{R}-$endomorphisms of the sheaf of graded algebras $\Omega_G(\mathcal{A})$. We say that $\Delta \in \mathrm{End}_\mathbb{R}(\Omega_G(\mathcal{A}))$ has bidegree (Δ_1, Δ_2) if Δ maps $\Omega_G^{(p,q)}(\mathcal{A})$ to $\Omega_G^{(p+\Delta_1, q+\Delta_2)}$, and if this is the case we write $\Delta \in \mathrm{End}_\mathbb{R}^{(\Delta_1, \Delta_2)}(\Omega_G(\mathcal{A}))$.

On the space $\mathrm{End}_\mathbb{R}(\Omega_G(\mathcal{A}))$ we introduce a bracket $[.,.]$ (called graded commutator) by means of

$$[\Delta, \Gamma] = \Delta \circ \Gamma - (-1)^{\langle \Delta, \Gamma \rangle} \Gamma \circ \Delta \in \mathrm{End}_\mathbb{R}^{(\Delta_1 + \Gamma_1, \Delta_2 + \Gamma_2)},$$

where $\langle \Delta, \Gamma \rangle = \Delta_1 \Gamma_1 + \Delta_2 \Gamma_2$ and it is easy to prove that this bracket turns $(\mathrm{End}_\mathbb{R}(\Omega_G(\mathcal{A})), [.,.])$ into a bigraded Lie algebra. This implies, among other things, that

$$[\Delta, \Gamma] = -(-1)^{\langle \Delta, \Gamma \rangle} [\Gamma, \Delta]$$

and the graded Jacobi identity

$$[\Delta, [\Gamma, \Lambda]] = [[\Delta, \Gamma], \Lambda] + (-1)^{\langle \Delta, \Gamma \rangle} [\Gamma, [\Delta, \Lambda]]. \tag{2.3}$$

Definition 7. *A differential operator on $\Omega_G(\mathcal{A})$ of bidegree (Δ_1, Δ_2) and order equal to or less than q, is an endomorphism $\Delta \in \mathrm{End}_\mathbb{R}^{(\Delta_1, \Delta_2)}(\Omega_G(\mathcal{A}))$ such that*

$$[[...[[\Delta, \mu_{a_0}], \mu_{a_1}], ...], \mu_{a_q}] = 0$$

for all $\mu_{a_i}, i \in \{0, ..., q\}$, where $a_i \in \Omega_G(\mathcal{A})$ and μ_{a_i} denotes the morphism multiplication by $a_i, \mu_{a_i}(b) = a_i \wedge b$ ($\forall b \in \mathcal{A}$), which has order 0 and bidegree that of a_i. The space of such operators is denoted $\mathcal{D}if_q^{(\Delta_1, \Delta_2)}(\Omega_G(\mathcal{A}))$ or even $\mathcal{D}if_q^{(\Delta_1, \Delta_2)}(\mathcal{A})$ if there is no risk of confusion.

We will restrict ourselves to operators vanishing on the identity, that is $\Delta(\mathbf{1}) = 0$.

Example 1. *The first-order differential operators are the derivations: if $\Delta \in \mathcal{D}if_1^{(\Delta_1, \Delta_2)}(\mathcal{A})$, then*

$$[[\Delta, \mu_{a_0}], \mu_{a_1}] = 0$$

is equivalent to $\Delta(a_0 \cdot a_1) = \Delta(a_0) \cdot a_1 + (-1)^{\langle \Delta, a_0 \rangle} a_0 \cdot \Delta(a_1)$. *Other examples are the known insertion operator* $i_A \in \mathcal{D}if_p^{(-p,|A|)}(\mathcal{A})$ *(where A is a graded $p-$multivector and $|A|$ its $\mathbb{Z}-$degree), the generalized Lie derivative* $\mathcal{L}_A^G = [i_A, \mathrm{d}^G] \in \mathcal{D}if_p^{(-p+1,|A|)}(\mathcal{A})$ *and* $\mathrm{d}^G \in \mathcal{D}if_p^{(1,0)}(\mathcal{A})$.

A useful result states that

$$[\mathcal{D}if_q^{(p_1,p_2)}(\mathcal{A}), \mathcal{D}if_{q'}^{(p_1',p_2')}(\mathcal{A})] \subset \mathcal{D}if_{q+q'-1}^{(p_1+p_1',p_2+p_2')}(\mathcal{A}), \tag{2.4}$$

and from that formula the following is easily proved:

Proposition 1. *Any differential operator* $\Delta \in \mathcal{D}if_\delta^{(\Delta_1,\Delta_2)}(\mathcal{A})$ *is determined by its action on graded forms with cohomological degree equal to or less than the order of the operator, that is, by* $\Delta|_{\Omega_G^{(p,q)}(\mathcal{A})}$ *where* $0 \leq p \leq \delta$.

Proposition 2. *Any operator* $\Delta \in \mathcal{D}if_q^{(-q,|A|)}(\mathcal{A})$ *has the form* i_A, *for some* $A \in A_G^q(\mathcal{A})$.

This is a consequence of (2.4) and the preceding Proposition. Another very useful algebraic property guarantees that, if $\Delta \in \mathcal{D}if_q^{(-\Delta_1,\Delta_2)}(\mathcal{A})$ with $\Delta_1 > q$, then $\Delta \equiv 0$. Indeed, by definition

$$\Delta : \Omega_G^{(p,k)}(\mathcal{A}) \to \Omega_G^{(p-\Delta_1,k+\Delta_2)}(\mathcal{A}),$$

and applying Proposition 1, Δ is determined by its action on $\Omega_G^{(p,k)}(\mathcal{A})$ with $0 \leq p \leq q$. But for these values, as $\Delta_1 > q$, we have $p - \Delta_1 < 0$ and consequently $\Delta = 0$.

According to (2.4), if we have two graded multivector fields $A \in A_G^{(A_1,A_2)}(\mathcal{A})$, $B \in A_G^{(B_1,B_2)}(\mathcal{A})$ then

$$[\mathcal{L}_A^G, \iota_B] \in [\mathcal{D}if_{A_1}^{(-A_1+1,A_2)}(\mathcal{A}), \mathcal{D}if_{B_1}^{(-B_1,B_2)}(\mathcal{A})] \subset \mathcal{D}if_{A_1+B_1-1}^{(-(A_1+B_1-1),A_2+B_2)}(\mathcal{A}),$$

and Proposition 2 enables us to give the following definition:

Definition 8. *The graded Schouten-Nijenhuis bracket of A and B, which will be denoted* $[A, B]_{GSN}$, *is the graded multivector field given by*

$$[\mathcal{L}_A^G, \iota_B] = \iota_{[A,B]_{GSN}}.$$

Remark 5. *Note that* $[A,B]_{GSN} \in A_G^{(A_1+B_1-1,A_2+B_2)}(\mathcal{A})$. *Also, it is worth to observe that the graded Schouten-Nijenhuis bracket has some features which are absent in the classical Schouten-Nijenhuis bracket. For instance, when we take a pair of graded $2-$vectors A, B one of which has even ghost degree, then* $[A,B]_{GSN} = 0$ *trivially (see 1 in Proposition 4 below).*

As an inmediate consequence of the graded Jacobi identity (2.3) we have

Proposition 3. *With the preceding notations*

$$[\mathcal{L}_A^G, \mathcal{L}_B^G] = \mathcal{L}_{[A,B]_{GSN}}^G.$$

Also, it is easy to verify other interesting properties of this bracket, namely: graded anticommutativity, graded Jacobi identity (with which it has the structure of bigraded Lie algebra) and graded Leibniz rule. The proofs are simple modifications of those of [Mic 87] and will be omitted.

Proposition 4. *With the preceding notations*

(1)

$$[A, B]_{GSN} = -(-1)^{(A_1-1)(B_1-1)+A_2 B_2}[B, A]_{GSN}.$$

(2)

$$\begin{aligned}[A, [B, C]_{GSN}]_{GSN} = {}&[[A, B]_{GSN}, C]_{GSN} \\ &+ (-1)^{(A_1-1)(B_1-1)+A_2 B_2}[B, [A, C]_{GSN}]_{GSN}.\end{aligned}$$

(3)

$$[A, B \wedge C]_{GSN} = [A, B \wedge C]_{GSN} \wedge C + (-1)^{(A_1-1)B_1+A_2 B_2} B \wedge [A, C]_{GSN}.$$

Thus, $(A_G(\mathcal{A}), [\_, \_]_{GSN})$ has the structure of a bigraded Gerstenhaber algebra.

We can characterize the action of $[\_, \_]_{GSN}$ by giving it only on the generators of $A_G(\mathcal{A})$:

$$\begin{cases} \{0\} = A_G^{-1}(\mathcal{A}) \ni [a, b]_{GSN} = 0, & \forall a, b \in \mathcal{A} \\[2mm] \mathcal{A} = A_G^0(\mathcal{A}) \ni [X, a]_{GSN} = X(a), & \forall a \in \mathcal{A}, \forall X \in A_G^1(\mathcal{A}) \\[2mm] A_G^1(\mathcal{A}) \ni [X, Y]_{GSN} = [X, Y], & \forall X, Y \in A_G^1(\mathcal{A}), \end{cases} \qquad (2.5)$$

where $[X, Y]$ denotes the bracket on $\mathrm{End}_{\mathbb{R}}(\Omega_G(\mathcal{A}))$.

We have mentioned the bigraded Gerstenhaber algebras. Let us recall some relevant definitions (cfr. [WGS 01]).

Definition 9. *Let $\mathcal{B} = \sum\limits_{b_1, b_2 \in \mathbb{Z}} \mathcal{B}_{b_1}^{b_2}$ be a bigraded commutative algebra over a commutative ring $\mathcal{R}$. A bigraded Lie algebra structure $[\_, \_]$, of bidegree $(-1, 0)$, on $\mathcal{B} = \sum\limits_{b_1, b_2 \in \mathbb{Z}} \mathcal{B}_{b_1}^{b_2}$ is called a bigraded Gerstenhaber algebra bracket if, for each $B \in \mathcal{B}_{b_1}^{b_2}$, $[B, \_]$ is a derivation of bidegree $(b_1 - 1, b_2)$ of $\mathcal{B} = \sum\limits_{b_1, b_2 \in \mathbb{Z}} \mathcal{B}_{b_1}^{b_2}$. In this case, $\mathcal{B}$ is called a bigraded Gerstenhaber algebra.*

Remark 6. *A bigraded Gerstenhaber algebra is also called a bigraded Poisson bracket of bidegree $(-1, 0)$. The general definition is the following.*

Definition 10. *A bigraded Poisson bracket on $\mathcal{B}$, with bidegree $|P| = (p_1, p_2)$, is an $\mathcal{R}-$bilinear map $\{\_, \_\} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ such that:*

(1) $|\{B_1, B_2\}| = |B_1| + |B_2| + |P|$, $\forall B_1, B_2 \in \mathcal{B}$.
(2) $\{B_1, B_2\} = -(-1)^{\langle B_1+P, B_2+P \rangle}\{B_2, B_1\}$.
(3) $\{B_1, B_2 \cdot B_3\} = \{B_1, B_2\} \cdot B_3 + (-1)^{\langle B_1+P, B_2 \rangle} B_2 \cdot \{B_1, B_3\}$.
(4) $\{B_1, \{B_2, B_3\}\} = \{\{B_1, B_2\}, B_3\} + (-1)^{\langle B_1+P, B_2+P \rangle}\{B_2, \{B_1, B_3\}\}$

$\forall B_1, B_2, B_3 \in \mathcal{B}$.

3. The Krasil'shchik-Schouten bracket on supermanifolds

This Section presents in a condensed form some of the results contained in [Kra 91].

Let $(M, \mathcal{A})$ be a supermanifold, with $\mathcal{A}$ a sheaf of graded commutative algebras over a commutative ring $\mathcal{R}$. We will write $a \cdot b = (-1)^{ab} b \cdot a$ for the product of sections of $\mathcal{A}$ (the superfunctions on $(M, \mathcal{A})$).

Now, consider the algebra of multiderivations on $\mathcal{A}$. This is the bigraded algebra

$$\mathcal{D}(\mathcal{A}) = \sum_{i \in \mathbb{Z}} \mathcal{D}_i(\mathcal{A}),$$

where

$$\mathcal{D}_{-i}(\mathcal{A}) = \{0\} \ , \ \forall i \in \mathbb{N}$$
$$\mathcal{D}_0(\mathcal{A}) = \mathcal{A}$$
$$\mathcal{D}_1(\mathcal{A}) = \{F \in \operatorname{Hom}_{\mathcal{R}}(\mathcal{A}, \mathcal{D}_0(\mathcal{A})) : F(a \cdot b) = F(a) \cdot b + (-1)^{aF} a \cdot F(b)\}$$
$$\cdots$$

$$\mathcal{D}_i(\mathcal{A}) = \{F \in \operatorname{Hom}_{\mathcal{R}}(\mathcal{A}, \mathcal{D}_{i-1}(\mathcal{A})) : F(a \cdot b) = F(a) \cdot b + (-1)^{aF} a \cdot F(b)$$
$$\text{and } (F(a))(b) = -(-1)^{ab}(F(b))(a)\},$$

(in $(-1)^{aF}$ we understand the degree of F as a morphism of $\mathbb{Z}-$graded algebras) and the algebra product is the mapping $* : \mathcal{D}_i(\mathcal{A}) \times \mathcal{D}_j(\mathcal{A}) \to \mathcal{D}_{i+j}(\mathcal{A})$ defined as follows:

(1) If $a, b \in \mathcal{D}_0(\mathcal{A}) = \mathcal{A}$, then $a * b = a \cdot b$.
(2) If $a \in \mathcal{D}_0(\mathcal{A}), F \in \mathcal{D}_1(\mathcal{A})$, then $a * F \in \mathcal{D}_1(\mathcal{A})$ is given by

$$a * F = a \cdot F = -(-1)^{aF} F * a,$$

where $a \cdot F : \mathcal{A} \to \mathcal{A}$ acts as $(a \cdot F)(b) = a \cdot (F(b)), \forall b \in \mathcal{A}$.
(3) Recursively, if $F \in \mathcal{D}_f(\mathcal{A}), G \in \mathcal{D}_g(\mathcal{A})$, then $F * G \in \mathcal{D}_{f+g}(\mathcal{A})$ acts as

$$(F * G)(a) = F * G(a) + (-1)^{aG+g} F(a) * G \ , \ \forall a \in \mathcal{A}. \tag{3.1}$$

Remark 7. *Clearly, for each $i \in \mathbb{Z}$, $\mathcal{D}_i(\mathcal{A})$ is an $\mathcal{A}-$module. By definition, given $a \in \mathcal{A}$ we have $F(a) \in \mathcal{D}_{f-1}(\mathcal{A}), G(a) \in \mathcal{D}_{g-1}(\mathcal{A})$, so (assuming the product $*$ defined on $\mathcal{D}_{f-1}(\mathcal{A}) \times \mathcal{D}_g(\mathcal{A})$ and $\mathcal{D}_f(\mathcal{A}) \times \mathcal{D}_{g-1}(\mathcal{A})$), formula (3.1) yields an element of $\mathcal{D}_{f+g}(\mathcal{A})$.*

Proposition 5. *The product $* : \mathcal{D}_i(\mathcal{A}) \times \mathcal{D}_j(\mathcal{A}) \to \mathcal{D}_{i+j}(\mathcal{A})$ is associative, distributive with respect to addition and it verifies*

$$F * G = (-1)^{FG+fg} G * F$$

when $F \in \mathcal{D}_f(\mathcal{A}), G \in \mathcal{D}_g(\mathcal{A})$.

Note that $\mathcal{D}(\mathcal{A}) = \sum_{i \in \mathbb{Z}} \mathcal{D}_i(\mathcal{A})$ has a grading inherited from that of $\mathcal{A}$. Moreover, we can consider on it another additional degree, given by the $f \in \mathbb{Z}$ to which an $F \in \mathcal{D}(\mathcal{A}) = \sum_{f \in \mathbb{Z}} \mathcal{D}_f(\mathcal{A})$ belongs, thus getting what is called a partitioned bigraded algebra with degree indexes (f, F) and marked degree f. So, we will write $|F| = (f, F)$ when $F \in \mathcal{D}_f(\mathcal{A})$.

Let us explain why the elements of $\mathcal{D}(\mathcal{A})$ are called multiderivations. This result will be important later.

Proposition 6. *For each $i \in \mathbb{Z}$, there exists an isomorphism*

$$\varphi : A_G^i(\mathcal{A}) = \mathrm{Hom}_{\mathcal{A}}(\Omega_G^i(\mathcal{A}), \mathcal{A}) \to \mathcal{D}_i(\mathcal{A}).$$

Proof. First, note that

$$\mathcal{D}_1(\mathcal{A}) = \{F \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{A}, \mathcal{A}) : F(a \cdot b) = F(a) \cdot b + (-1)^{ab} a \cdot F(b)\}$$
$$= \mathrm{Der}(\mathcal{A}) = A_G^1(\mathcal{A}).$$

Let us assume $\mathcal{D}_i(\mathcal{A}) \simeq A_G^i(\mathcal{A})$ for $i \in \mathbb{N}$; then, if $F \in \mathcal{D}_{i+1}(\mathcal{A})$ we have that $F : \mathcal{A} \to \mathcal{D}_i(\mathcal{A})$ verifies

$$F(a \cdot b) = F(a) \cdot b + (-1)^{aF} a \cdot F(b),$$

so $F \in \mathcal{D}_1(\mathcal{A}) \otimes \mathcal{D}_i(\mathcal{A}) \simeq A_G^1(\mathcal{A}) \otimes A_G^i(\mathcal{A})$. Moreover, (by definition of $\mathcal{D}_i(\mathcal{A})$),

$$(F(a))(b) = -(-1)^{ab}(F(b))(a),$$

thus, actually $F \in A_G^1(\mathcal{A}) \wedge A_G^i(\mathcal{A})$.

Reciprocally, if $A \wedge B \in A_G^{i+1}(\mathcal{A}) = A_G^1(\mathcal{A}) \wedge A_G^i(\mathcal{A})$ is a decomposable graded multivector field, we define the mapping

$$F_{A \wedge B} : \mathcal{A} \to \mathcal{D}_i(\mathcal{A}) \simeq A_G^i(\mathcal{A})$$

through

$$F_{A \wedge B}(a) = A \wedge B(a) + (-1)^{ab} A(a) \cdot B,$$

which verifies (this is an easy computation) $F_{A \wedge B}(a \cdot b) = F(a) \cdot b + (-1)^{a(A+B)} a \cdot F_{A \wedge B}(b)$ and $(F_{A \wedge B}(a))(b) = -(-1)^{ab}(F_{A \wedge B}(b))(a)$.

Finally, note that if we start from $A \wedge B \in A_G^i(\mathcal{A})$, consider $F_{A \wedge B} \in \mathcal{D}_i(\mathcal{A})$ and then return to $A_G^i(\mathcal{A})$ applying the preceding procedure, we will arrive at the element $A \wedge B$. The same thing happens following the reverse path, so we have an isomorphism $A_G^i(\mathcal{A}) \simeq \mathcal{D}_i(\mathcal{A})$. $\qquad \square$

Remark 8. *As a consequence of this result, note that the marked degree in $\mathcal{D}(\mathcal{A}) = A_G(\mathcal{A})$ is the cohomological degree.*

In addition to the product $*$, we can define another structure on $\mathcal{D}(\mathcal{A})$, a bracket $[\![\_, \_]\!]$ called the Krasil'shchik-Schouten bracket. It will be constructed recursively in such a way that $[\![\_, \_]\!] : \mathcal{D}_i(\mathcal{A}) \times \mathcal{D}_j(\mathcal{A}) \to \mathcal{D}_{i+j-1}(\mathcal{A})$.

(1) If $a, b \in \mathcal{D}_0(\mathcal{A}) = \mathcal{A}$, then

$$[\![a, b]\!] = 0. \tag{3.2}$$

(2) If $a \in \mathcal{D}_0(\mathcal{A}), F \in \mathcal{D}_1(\mathcal{A})$, then

$$[\![F, a]\!] = F(a) = (-1)^{aF+f}[\![a, F]\!]. \tag{3.3}$$

(3) Recursively, if $F \in \mathcal{D}_f(\mathcal{A}), G \in \mathcal{D}_g(\mathcal{A})$, then the action of $[\![F, G]\!] : \mathcal{A} \to \mathcal{D}_{f+g-2}(\mathcal{A})$ is given by

$$[\![F, G]\!](a) = [\![F, G(a)]\!] + (-1)^{FG+g}[\![F(a), G]\!] \,, \, \forall a \in \mathcal{A}. \tag{3.4}$$

The main properties of $[\![\_, \_]\!]$ are gathered in the following result (due to Krasil'shchik, [Kra 91]).

Proposition 7. *The mapping $[\![\_, \_]\!] : \mathcal{D}_i(\mathcal{A}) \times \mathcal{D}_j(\mathcal{A}) \to \mathcal{D}_{i+j-1}(\mathcal{A})$ is $\mathcal{R}-$bilinear and it verifies:*

(1)

$$[\![F, G]\!] = -(-1)^{FG+(f-1)(g-1)}[\![G, F]\!].$$

(2)
$$[\![F, G * H]\!] = [\![F, G]\!] * H + (-1)^{FG + (f-1)g} G * [\![F, H]\!].$$

(3)
$$[\![F, [\![G, H]\!]]\!] = [\![[\![F, G]\!], H]\!] + (-1)^{FG + (f-1)(g-1)} [\![G, [\![F, H]\!]]\!].$$
$$\forall F \in \mathcal{D}_f(\mathcal{A}), G \in \mathcal{D}_g(\mathcal{A}), H \in \mathcal{D}_h(\mathcal{A}).$$

Taking into account that $\mathcal{D}(\mathcal{A})$ is a marked bigraded algebra, we see that $[\![\_, \_]\!]$ is an example of marked bigraded Poisson bracket on $\mathcal{D}(\mathcal{A})$, with marked bidegree $(-1, 0)$ (the -1 takes place in the marked index), that is to say, $(\mathcal{D}(\mathcal{A}), [\![\_, \_]\!])$ is a bigraded Gerstenhaber algebra.

The Krasil'shchik-Schouten bracket is a tool which unifies many of the objects encountered in the theory of calculus over modules (this was indeed one of the reasons Krasil'shchik introduced it in [Kra 91]). For multivectors on supermanifolds it coincides with the graded Schouten-Nijenhuis bracket of Paragraph 2.4 (see the next Example), but the construction of Krashil'shchick is far more general and it applies to the case where $\mathcal{A}$ is a sheaf of $k-$graded commutative algebras.

Example 2 (the graded Schouten-Nijenhuis bracket). *Consider $M \supset U \mapsto \mathcal{A}(U)$ a sheaf of graded commutative algebras. We know that the sheaves of multiderivations on $\mathcal{A}$, $\mathcal{D}(\mathcal{A})$, are naturally isomorphic to the sheaves of graded multivectors $A_G(\mathcal{A})$, where we have defined the graded Schouten-Nijenhuis bracket $[\_, \_]_{GSN}$ (see 2.4). This isomorphism translates $[\![\_, \_]\!]$ into $[\_, \_]_{GSN}$, more precisely*

Proposition 8. $\varphi : (A_G^i(\mathcal{A}), [\_, \_]_{GSN}) \to (\mathcal{D}_i(\mathcal{A}), [\![\_, \_]\!])$ *is an isomorphism of bigraded Gerstenhaber algebras.*

Proof. To see this, note that the expressions (3.2), (3.3), (3.4) reduce to (2.5) when we consider the generators of $A_G^i(\mathcal{A})$: $a \in \mathcal{A} = A_G^0(\mathcal{A})$ and $X \in A_G^1(\mathcal{A})$. $\qquad \square$

Example 3. *In particular, if we choose $\mathcal{A}(U) = C^\infty(U)$ (that is, the trivial sheaf of differentiable functions on M), then $\mathcal{D}(C^\infty(U)) \simeq A(C^\infty(U)) = \Lambda_{C^\infty(U)}(TU)$, that is, multiderivations are usual multivectors, and $[\![\_, \_]\!]$ is nothing but the usual Schouten-Nijenhuis bracket.*

4. Divergences and generators of the Krashil'shchick-Schouten bracket

As we mentioned before, the Krasil'shchik-Schouten bracket can be viewed as a (marked bigraded) Poisson bracket on $\mathcal{D}(\mathcal{A})$, so it makes sense to ask for its generating operators and how to construct them. Let us recall the basic concepts.

Definition 11. *Let $(\mathcal{B}, \{\_, \_\})$ be a bigraded Poisson structure, that is, $\{\_, \_\}$ a bigraded Poisson bracket as in Definition 10 with bidegree P. A second-order differential operator on $\mathcal{B}$, $\Delta \in \mathcal{D}if_2(\mathcal{B})$, is called a generating operator for the bracket $\{\_, \_\}$ if, for all $B_1, B_2 \in \mathcal{B}$, we can express $\{B_1, B_2\}$ as*

$$\{B_1, B_2\} = (-1)^{\langle B_1, P \rangle}(\Delta(B_1 \cdot B_2) - \Delta(B_1) \cdot B_2 - (-1)^{\langle B_1, P \rangle} B_1 \cdot \Delta(B_2)). \quad (4.1)$$

In the following, we will consider the case $\mathcal{B} = \mathcal{D}(\mathcal{A}) = A_G(\mathcal{A})$ with the algebra structure given by the product $\wedge$ on graded multivectors. We show that, as in the classical theory, generating operators of the bigraded Gerstenhaber structure on $\mathcal{D}(\mathcal{A})$ can be obtained with the aid of divergence operators.

Definition 12. *A divergence operator on $\mathcal{A}$ is a linear morphism of sheaves of $\mathbb{R}-$vector spaces*

$$\begin{aligned} \mathrm{div} : \mathrm{Der}(\mathcal{A}) &\to \mathcal{A} \\ D &\mapsto \mathrm{div}(D) \end{aligned} \qquad (4.2)$$

such that $\forall a \in \mathcal{A}$,

$$\mathrm{div}(a \cdot D) = a \cdot \mathrm{div}(D) + (-1)^{aD} D(a).$$

Examples of divergence operators can be obtained from the Berezinian sheaf of a supermanifold and their associated Berezinian volume elements, or from graded connections. We refer the reader to [Kos-Mon 02] for details.

The main result is then the following.

Theorem 1. *Let* div *be a divergence operator on $\mathcal{A}$. Then, $-$div can be uniquely extended into a generating operator Δ_{div} of bidegree $(-1,0)$ of the bigraded Gerstenhaber algebra $(A_G(\mathcal{A}) = \Lambda_{\mathcal{A}} \mathrm{Der}(\mathcal{A}), [\_,\_]_{GSN})$, and Δ_{div} commutes (in the graded sense) with the insertion of the graded $1-$form $d^G a$, $\iota_{d^G a}$, for any $a \in \mathcal{A}$.*

Proof. Let us write Δ_{div} simply as Δ. First, note that for any $a \in \mathcal{A}$ and graded multivectors $A, B \in A_G(\mathcal{A})$:

$$\iota_{d^G a}(A \wedge B) = \iota_{d^G a}(A) \wedge B + (-1)^{A_1 + A_2 a} A \wedge \iota_{d^G a}(B) \qquad (4.3)$$

and

$$\iota_{d^G a}[A, B]_{GSN} = [\iota_{d^G a}A, B]_{GSN} + (-1)^{A_1 - 1 + A_2 a}[A, \iota_{d^G a}B]_{GSN}. \qquad (4.4)$$

Now, because its cohomological degree must be -1, the operator Δ has to vanish on $A_G^0(\mathcal{A}) \simeq \mathcal{A}$ and, because we want it to be an extension, it has to coincide with $-$div on $A_G^1(\mathcal{A}) \simeq \mathrm{Der}(\mathcal{A})$:

$$\begin{aligned} \Delta|_{A_G^0(\mathcal{A})} &\doteq 0 \\ \Delta|_{A_G^1(\mathcal{A})} &\doteq -\mathrm{div}. \end{aligned}$$

Assume that it is defined on graded multivectors A of bidegree (A_1, A_2) with cohomological degree $A_1 \leq k$, that

$$\iota_{d^G a}(\Delta(A)) = -\Delta(\iota_{d^G a}A) \qquad (4.5)$$

for all $a \in \mathcal{A}$ and $A \in A_G(\mathcal{A})$ with $A_1 \leq k$, and that $\Delta(A)$ satisfies the defining relation (4.1) for generators of the bracket $[\_,\_]_{GSN}$ for all graded multivectors A and B such that $A_1 + B_1 \leq k$ (sum of cohomological degrees). Relation (4.5) then defines $\Delta(C)$ for graded multivectors C of cohomological degree $k+1$. Let us prove that (4.1) is satisfied for $A_1 + B_1 \leq k+1$.

Indeed, let $A, B \in A_G(\mathcal{A})$ be as above ($A_1 + B_1 \leq k$) and let $X \in A_G^1(\mathcal{A})$, so $A \wedge X \wedge B$ has cohomological degree $k+1$. Using (4.4) we write

$$\iota_{d^G a}[A, X \wedge B]_{GSN} = [\iota_{d^G a}A, X \wedge B]_{GSN} + (-1)^{A_1 - 1 + A_2 a}[A, \iota_{d^G a}(X \wedge B)]_{GSN}.$$

Let us study the right hand side. Since the sum of the cohomological degrees of $\iota_{d^G a}A$ and $X \wedge B$ is k, applying the induction hypothesis:

$$\begin{aligned} [\iota_{d^G a}A, X \wedge B]_{GSN} = (-1)^{A_1 - 1} \big(&\Delta\left(\iota_{d^G a}A \wedge X \wedge B\right) \\ &- \Delta\left(\iota_{d^G a}A\right) \wedge X \wedge B - (-1)^{A_1 - 1} \iota_{d^G a}A \wedge \Delta\left(X \wedge B\right) \big), \end{aligned}$$

and

$$(-1)^{A_1-1+A_2a}\left[A, \iota_{d^G a}\left(X \wedge B\right)\right]_{GSN} = -(-1)^{A_2a}\left\{\Delta\left(A \wedge \iota_{d^G a}\left(X \wedge B\right)\right)\right.$$
$$- \Delta\left(A\right) \wedge \iota_{d^G a}\left(X \wedge B\right)$$
$$\left.-(-1)^{A_1} A \wedge \Delta\left(\iota_{d^G a}\left(X \wedge B\right)\right)\right\},$$

so we get, using (4.5):

$$\iota_{d^G a}\left[A, X \wedge B\right]_{GSN} = (-1)^{A_1-1}\Delta\left(\iota_{d^G a}A \wedge X \wedge B + (-1)^{A_1+A_2a} A \wedge \iota_{d^G a}\left(X \wedge B\right)\right)$$
$$- (-1)^{A_1}\iota_{d^G a}\Delta\left(A\right) \wedge X \wedge B - \iota_{d^G a}A \wedge \Delta\left(X \wedge B\right)$$
$$+ (-1)^{A_2a}\Delta\left(A\right) \wedge \iota_{d^G a}\left(X \wedge B\right)$$
$$- (-1)^{A_1+A_2a} A \wedge \iota_{d^G a}\Delta\left(X \wedge B\right),$$

thus, by (4.3) and (4.5) again:

$$\iota_{d^G a}\left[A, X \wedge B\right]_{GSN} = (-1)^{A_1-1}\Delta\left(\iota_{d^G a}\left(A \wedge X \wedge B\right)\right)$$
$$+ (-1)^{A_1-1}\iota_{d^G a}\left(\Delta\left(A\right) \wedge X \wedge B\right)$$
$$- \iota_{d^G a}\left(A \wedge \Delta\left(X \wedge B\right)\right)$$
$$= (-1)^{A_1}\iota_{d^G a}\left(\Delta\left(A \wedge X \wedge B\right)\right.$$
$$- \Delta\left(A\right) \wedge X \wedge B$$
$$\left.-(-1)^{A_1} A \wedge \Delta\left(X \wedge B\right)\right).$$

That is, for all $a \in \mathcal{A}$:

$$\iota_{d^G a}\left[A, X \wedge B\right]_{GSN} = \iota_{d^G a}\left((-1)^{A_1}\Delta\left(A \wedge X \wedge B\right)\right.$$
$$- \Delta\left(A\right) \wedge X \wedge B$$
$$\left.-(-1)^{A_1} A \wedge \Delta\left(X \wedge B\right)\right),$$

or

$$\left[A, X \wedge B\right]_{GSN} = (-1)^{A_1}\left(\Delta\left(A \wedge X \wedge B\right) - \Delta\left(A\right) \wedge X \wedge B - (-1)^{A_1} A \wedge \Delta\left(X \wedge B\right)\right).$$

$\square$

Corollary 1. *Let* div *be a divergence operator on* $\mathcal{A}$*. Then,* $-$div *can be uniquely extended into a generating operator* Δ_{div} *of bidegree* $(-1, 0)$*, of the bigraded Gerstenhaber algebra* $\mathcal{D}(\mathcal{A})$ *endowed with the Krashil'shchick-Schouten bracket* $[\![\_, \_]\!]$*.*

Proof. This follows from the isomorphism defined in Example 2. $\square$

5. Some comments on the Jacobi case

In a very recent paper, J. Grabowski and G. Marmo (see [Gra-Mar 03]) have extended the Krasil'shchik-Schouten bracket to a $(n + 1)-$graded Jacobi bracket, this time defined on the commutative algebras of multidifferential operators of first order (notice the change in the terminology: they speak about polydifferential operators). The graded multiderivations we are considering in this paper are a graded subspace of that of multidifferential operators[1] of first order, and are characterized by the fact that they kill the unit element, that is, $D(\mathbf{1}) = 0$ for any graded multiderivation. The direct consequence of this fact is that, for the $(n+1)-$graded Jacobi

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[1]See equation (92) in [Gra-Mar 03]. Note that they use $\mathcal{D}^\alpha$ not for multiderivations, as we do, but for multidifferential operators.

bracket, the Leibniz rule (item (2) in Proposition 7) is replaced by a generalized expression such as

$$[\![F, G * H]\!] = [\![F, G]\!] * H + (-1)^{FG+(f-1)g} G * [\![F, H]\!] - [\![F, \mathbf{1}]\!] * H * G. \qquad (5.1)$$

Now, the question whether the $m-$graded Jacobi brackets admit generating operators in the same way that the Krasil'shchik-Schouten bracket does, arises naturally. This is a very interesting question but it can not be answered by a straightforward generalization of the techniques used in this paper. This is easy to see even in the case considered by Koszul in [Koz 85] (the Schouten-Nijenhuis bracket), a particular instance of Krasil'shchik's construction.

Indeed, let $\mathcal{A} = \bigoplus_{p \geq 0} \mathcal{A}^p$ be a $\mathbb{Z}_2-$graded commutative algebra, so $ab = (-1)^{ab}ba$ for each $a \in \mathcal{A}^a, b \in \mathcal{A}^b$, and consider the $\mathbb{Z}_2-$graded Lie algebra of endomorphisms $\mathrm{End}(\mathcal{A}) = \bigoplus_{k \in \mathbb{Z}} \mathrm{End}^k(\mathcal{A})$, with the bracket as in Paragraph 2.4. Koszul defines, for each endomorphism D and $r \in \mathbb{Z}$, an $r-$form on $\mathcal{A}$ with values on $\mathcal{A}$, $\Phi_D^r(a_1, ..., a_r)$, such that for instance:

$$\Phi_D^1(a) = D(a) - D(\mathbf{1})a,$$
$$\Phi_D^2(a, b) = D(ab) - D(a)b - (-1)^{aD}aD(b) + D(\mathbf{1})ab,$$
$$\Phi_D^3(a, b, c) = D(abc) - D(a)bc - (-1)^{aD}aD(bc) - (-1)^{b(a+D)}bD(ac)$$
$$+ D(a)bc + (-1)^{aD}aD(b)c + (-1)^{D(a+b)}abD(c) - D(\mathbf{1})abc,$$

and it is easily verified that the following relation holds:

$$\Phi_D^3(a, b, c) = \Phi_D^2(a, bc) - \Phi_D^2(a, b)c - (-1)^{bc}\Phi_D^2(a, c)b. \qquad (5.2)$$

This is the embryo of the Leibniz rule for the Schouten-Nijenhuis bracket. Note that it is independent of the property $D(\mathbf{1}) = 0$.

The connection with the results presented here, rests upon realizing that the definition of Φ_D^k can be given in terms of the commutator of endomorphisms (with the notations of Definition 7) as

$$\Phi_D^k(a_1, ..., a_k) = [[[...[D, \mu_{a_1}], ...], \mu_{a_{k-1}}], \mu_{a_k}](\mathbf{1}).$$

Now, given a second-order differential operator D with odd degree $|D|$, Koszul defines the associated bracket on $\mathcal{A}$, $[\_, \_]_D$,

$$[a, b]_D = (-1)^a \Phi_D^2(a, b), \ \forall a, b \in \mathcal{A},$$

and it is immediate, from the explicit expressions for $\Phi_D^2(a, b)$ above, that it satisfies

$$[a, b]_D = -(-1)^{(a-D)(b+D)}[b, a]_D \qquad (5.3)$$

and

$$[a, bc]_D = [a, b]_D c + (-1)^{(a+D)b}b[a, c]_D, \qquad (5.4)$$

the last property being a consequence of (5.2) and the fact that D is second-order (so $\Phi_D^3(a, b, c) = 0$). If, in addition, $D(\mathbf{1}) = 0$ and D^2 is second-order, then $[\_, \_]_D$ also satisfies the Jacobi identity.

A bracket on $\mathcal{A}$ is said to have generating operator D if it is of the type $[\_, \_]_D$. But, as we see from (5.4), such a bracket derived from Φ_D^2 always gives a "Leibniz rule", and not a generalized expression such as (5.1). Thus, in order to speak about generating operators for $n-$graded Jacobi brackets along these lines, another (different) approach is needed.

However, there is still another way in which one can think of generating operators for $n-$graded Jacobi structures, and it has its foundations in some results presented in [Gra-Mar 03]: concretely Theorem 12 and Corollary 5 in that reference, where it is proved that the space of multidifferential operators of first order can be decomposed into direct sums of subspaces of multiderivations.

For the case we are considering in this paper, where $\mathcal{A}$ is a graded $1-$algebra, the results of Grabowski and Marmo state the decomposition (in our notation!)

$$\mathcal{D}if_q(\mathcal{A}) = \mathcal{D}_q(\mathcal{A}) \oplus \cdots \oplus \mathcal{D}_0(\mathcal{A}),$$

where the Krasil'shchik-Schouten bracket $[\![\_, \_]\!]$ is defined on each $\mathcal{D}_i(\mathcal{A})$, $0 \leq i \leq q$. Then, applying Lemma 4 in [Gra-Mar 03] we can extend this bracket to $\mathcal{D}if_q(\mathcal{A})$. In this context, it makes sense to say that a generating operator for the Krasil'shchik-Schouten bracket is also a generating operator for the extended Jacobi-Krasil'shchik-Schouten bracket introduced by Grabowski and Marmo.

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Dept. Geometria i Topologia, Universitat de València (Spain), Postal Address: Av. V. A. Estellés 1, 46100 Burjassot., Facultad de Ciencias, Universidad Autónoma de San Luis Potosí (México), Postal Address: Lat. Av. Salvador Nava s/n, 78290 SLP.

E-mail address: `jose.a.vallejo@fc.uaslp.mx`